3.0 Transmission Line Theory

Since standard circuit theory cannot be employed on an electrical network at microwave frequencies, an alternative analysis must be applied to the system. Transmission line theory is a tool which bridges the gap between circuit theory and a complete field analysis [10, 34]. Transmission lines (t-lines) lie between a fraction of a wavelength and many wavelengths in size. Conversely, in circuit analysis, the physical dimensions of the network are much smaller than the wavelength. A t-line is considered a distributed parameter network as opposed to a circuit which consists of lumped elements. Consequently, the voltages and currents associated with a propagating wave in a t-line can vary in both phase and magnitude over the length of the line. A t-line is also characterized by a propagation constant, k, and a characteristic impedance, Z_0.

3.1 TEM Waveguides

A waveguide is a structure which supports the propagation of specific modes (TEM, TE, or TM). Accordingly, a transmission line that consists of two or more conductors may support TEM modes, which are characterized by the lack of longitudinal field components and whose fields have a static distribution along the propagation direction, $\hat{z}$, such that,

$$\nabla^2 \boldsymbol{E} = \nabla^2 \boldsymbol{H} = 0 \tag{31}$$

The propagation constant is then defined only in the direction of propagation

$$|\boldsymbol{k}| = k_z = \frac{2\pi}{\lambda_0}\sqrt{\varepsilon'} \tag{32}$$

TEM modes also have a uniquely defined voltage, current, and characteristic impedance. In general, the impedance for a traveling TEM mode (Z_{TEM}) is related to the permittivity of a material via the intrinsic impedance (η) of the medium in which it is propagating,

$$Z_{TEM} = \eta = \sqrt{\frac{\mu_0}{\varepsilon}} \tag{33}$$

3.1.1 Telegrapher's Equations

The Helmholtz equations, which yield the necessary plane wave solutions, can also be written in terms of voltage (V) and current (I); both measurable quantities [10]. Using a TEM analysis of a waveguide supporting voltage and current waves, Telegrapher's equations are derived as a series of two coupled equations,

$$\frac{\partial V}{\partial z} = -\frac{\partial}{\partial t}(LI) = -L\frac{\partial I}{\partial t} \tag{34}$$

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t} \tag{35}$$

Where the inductance and capacitance of the equivalent circuit are represented as L & C respectively. Now, if a harmonic time dependence is assumed, decoupling of Telegrapher's equations yields

$$\frac{\partial^2 V}{\partial z^2} = -\omega^2 LCV \tag{36}$$

$$\frac{\partial^2 I}{\partial z^2} = -\omega^2 LCI \tag{37}$$

The familiar plane wave solutions to these equations are in terms of both forward and backward traveling voltage and current waves given by,

$$V(z) = V_0^+ e^{-jkz} + V_0^- e^{jkz} \tag{38}$$

$$I(z) = I_0^+ e^{-jkz} + I_0^- e^{jkz} \tag{39}$$

3.1.2 Characteristic Impedance

The resulting voltage and current waves define the characteristic impedance of the t-line via the ratio of the voltage to current amplitudes,

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \sqrt{\frac{L}{C}} \tag{40}$$

This characteristic impedance is analogous to the intrinsic impedance, η, of a TEM plane wave. It is a property of the t-line and depends on the geometry of the line and the propagation medium. It does not represent ohmic power loss (i.e. loss due to charge motion or resistance), but rather accounts for the power flow down the t-line.

3.2 Measurable Quantities

In order to extract the desired dielectric and dimensional information from a sample using the reflection method, the relevant measurable quantities must be determined. In an electrical network, those quantities are the voltage and current waves as derived in the decoupled Telegrapher's equations. More specifically, the ratio of the reflected to incident voltage waves is measured in terms of a reflection coefficient, $\Gamma(z)$. These reflections are produced from the material interfaces or equivalently from transmission line loads.

In general for a normally incident microwave impinging on a dielectric material from free-space, the total energy of the incident beam will be divided between the reflected and transmitted waves. The proportion of energy division is determined by the impedance mismatch at the interface and hence depends on the permittivity of the mediums [11, 35]. For the free-space reflection method, the sample is backed by a conductor so that there is a complete reflection at the dielectric-conductor interface. The wave will then travel back through the dielectric sample and transmit through free-space to the transceiver antenna.

A comparison between the incident and reflected waves can be made to yield the measurable parameter Γ(z). This reflection coefficient is complex and carries both the phase and magnitude variations along the length of the sample. It is defined as,

$$\Gamma(z) \equiv \frac{V^-}{V^+} = \Gamma_0 e^{j2kz} \tag{41}$$

where Γ_0 represents the ratio of the magnitudes of the reflected to incident voltages,

$$\Gamma_0 = \frac{V_0^-}{V_0^+} \tag{42}$$

In a similar fashion, the impedance also undergoes spatial variation along the line and can be written in terms of Γ(z).

$$Z(z) = \frac{V(z)}{I(z)} = \left[\frac{1+\Gamma(z)}{1-\Gamma(z)}\right] Z_0 \tag{43}$$

3.2.1 Load Conditions

The impedance and reflection coefficient functions can be applied to a generally loaded section of transmission line in order to determine the input impedance at a specific position along the line. Figure 3.1 shows a diagram of a generally loaded t-line with characteristic impedance Z_0 and load impedance Z_L.

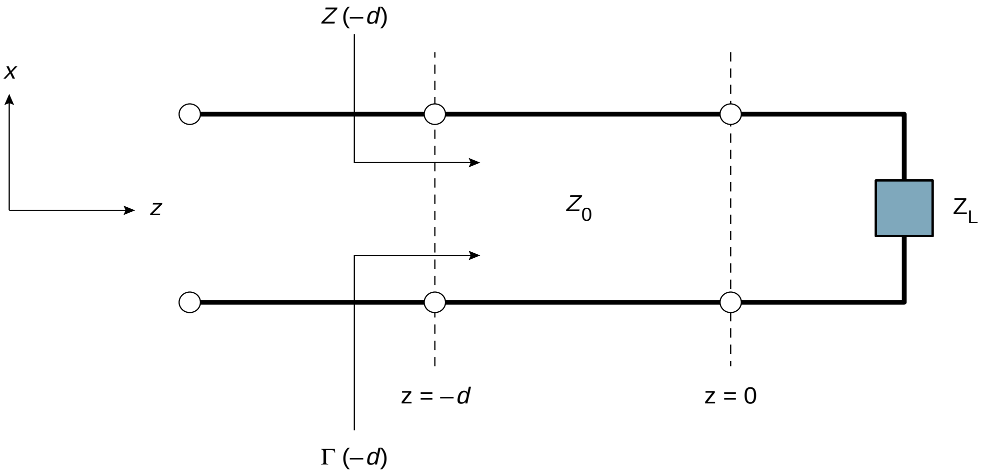

Figure 3.1: Schematic model of a generally loaded t-line

Analyzing the impedance at z = 0 via equation (43) yields a general function which determines the magnitude of the reflection coefficient, Γ_0

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{44}$$

At a position of z = -d, the reflection coefficient varies as,

$$\Gamma(-d) = \Gamma_0 e^{-j2kd} \tag{45}$$

Therefore, by substituting (44) and (45) into (43) and making a trigonometric substitution for the exponential function, a general expression is derived which yields the input impedance of the t-line/load network, Z(-d), for a line length of d, and with variational dependence on the load impedance, characteristic impedance, t-line thickness, and the permittivity of the t-line.

$$Z(-d) = Z_0\left[\frac{Z_L + jZ_0\tan(kd)}{Z_0 + jZ_L\tan(kd)}\right] \tag{46}$$

By investigating several different load possibilities and utilizing the periodicity of the tangent, this resulting input impedance variation can be separated into 4 special cases.

1) Matched Load

In this situation, the load at the end of the t-line has an impedance which is equal to the characteristic impedance, Z_0, of the line. As a result of this impedance match at the t-line/load interface, all of the power in the propagating microwave is transmitted to the load. In effect, there will be no reflection at the interface and $\Gamma_0 = 0$. This result can be mathematically noted by substituting $Z_L = Z_0$ into equation (46) and thus revealing that the line impedance equals the characteristic impedance.

$$Z(-d) = Z_0 \tag{47}$$

2) Short Circuit

As discussed previously, microwaves completely reflect from a highly conductive material. Thus, a short circuited load will completely reflect an incident microwave beam. This effect can be seen by substituting $Z_L = 0$ into (44), which yields a reflection coefficient of $\Gamma_0 = -1$. In terms of voltage and current, V = 0 at the load while the current, I, is a maximum there. Since the impedance of a short is zero, the resulting input impedance variation due to the shorted load will have a simple periodic variation,

$$Z(-d) = jZ_0 \tan(kd) \tag{48}$$

3) Open Line

For an open load, the impedance is infinite ($Z_L \rightarrow \infty$) and so the corresponding reflection coefficient at the load is $\Gamma_0 = 1$. In this case, the voltage is a maximum at the load and I = 0. Again using (46), the input impedance for an open line varies as,

$$Z(-d) = -jZ_0 \cot(kd) \tag{49}$$

4) Quarter-wavelength T-line (quarter-wave transformer)

The periodicity of the input impedance in (46) is also dependent on the length of the corresponding t-line. Since this relation has a periodicity of λ/2, then at positions of λ/4 along the line the tangent function will approach infinity and the resulting input impedance will vary as,

$$Z(d = \lambda/4) = \frac{Z_0^2}{Z_L} \tag{50}$$

An interesting relation between the short and open line cases becomes apparent by observing the variation of equation (48) for the two different load impedances. For a t-line with a shorted load ($Z_L = 0$), at a quarter-wavelength away from the short, the input impedance will approach infinity. Conversely, an open line ($Z_L = \infty$) observed at λ/4 away from the load will yield an input impedance of zero. Thus, a short circuit and an open line oscillate over every quarter-wavelength. This property is important when making calibration measurements. When using standard loads to calibrate a network analyzer, a short circuit can be used for both the shorted and open loads via an axial displacement.

With these relations for the input impedance along a loaded t-line in hand, an expression for the reflection coefficient, Γ(-d), as it varies over the length of the line can be derived from equation (44)

$$\Gamma(-d) = \frac{Z(-d) - Z_0}{Z(-d) + Z_0} \tag{51}$$

The reflection coefficient represents the ratio of the reflected to incident field in terms of the impedance variation in the t-line, and is the measurable quantity in the free-space microwave reflectometry apparatus.